Letters

Approximate Determination of the Characteristic Impedance of the Coaxial System Consisting of a Regular Polygon Concentric with a Circle

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Abstract—**This letter gives an approximate method for the determination of the characteristic impedance of an important family of coaxial systems. The approach makes use of the method of conformal mapping, and it has already been used in other technical applications by the first author and his associates.**

Riblet [1] has developed an interesting technique for the accurate determination of the characteristic impedance of the coaxial system consisting of a square concentric with a circle. An alternate procedure is presented here.

It has already been shown that for a certain class of doubly connected regions, the transformation onto a circular annulus can be obtained in a very simple manner if the mapping function for one of the boundaries is known [2]–[3].

Consider first a simply connected region with several axes of symmetry [Fig. 1(a)]. In general, the approximate mapping function which transforms this domain in the w plane onto a unit circle in the ξ plane is given by a functional relation of the form

$$w = \sum_{n=0}^{N} a_{1+np}\xi^{1+np}, \qquad \xi = re^{i\theta} \tag{1}$$

where p is the number of axes of symmetry of the configuration. Assume now a doubly connected region with the same external boundary and an inner concentric circular boundary [Fig. 1(b)] of radius R.

When $r < 1$ in (1), it is obvious that the first term is predominant since the exponents are given by $(1 + np)$, and for $n > 0$ the terms will decay rapidly. In other words, circles in the ξ plane will map into approximate circles in the w plane. The deviations become smaller as r decreases and also as a function of the number of axes of symmetry p. Take, for instance, the case of polygons of regular polygonal shape. In this case the mapping function is given by

$$w = a_p \cdot A_p \int_0^{\xi} \frac{du}{(1 - u^p)^{2/p}} \tag{2}$$

where a_p is the apothem of the polygon and A_p is a constant given in Table I.

Expressing (2) in series form, one obtains

$$w = a_p \cdot A_p \sum_{j=0}^{\infty} (-1)^j \binom{-\frac{2}{p}}{j} \frac{\xi^{jp+1}}{jp+1}. \tag{3}$$

In practice, a truncated series is used, and one then has

$$w = a_p \cdot A_p \sum_{j=0}^{J} a_j \cdot \xi^{jp+1} \tag{4}$$

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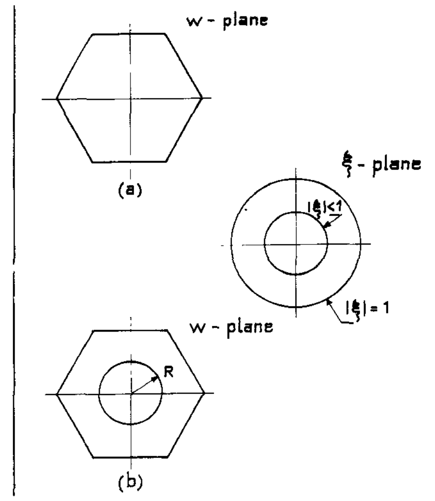
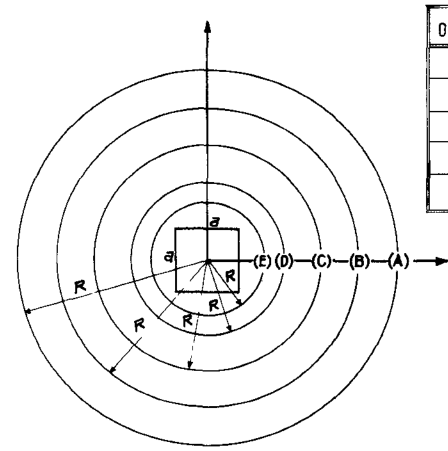


Fig. 1. Approximate mapping procedure for a certain type of doubly connected region.



Outer Boundary	λ	Max Error
(A)	5	0.03%
(B)	4	0.06%
(C)	3	0.2%
(D)	2	1.1%
(E)	1.5912	2.5%

Fig. 2. Circular waveguide with square inner boundary.

TABLE I
VALUES OF A_p

p	A_p
4	1.07870
5	1.05246
6	1.03754
7	1.02823
8	1.02201

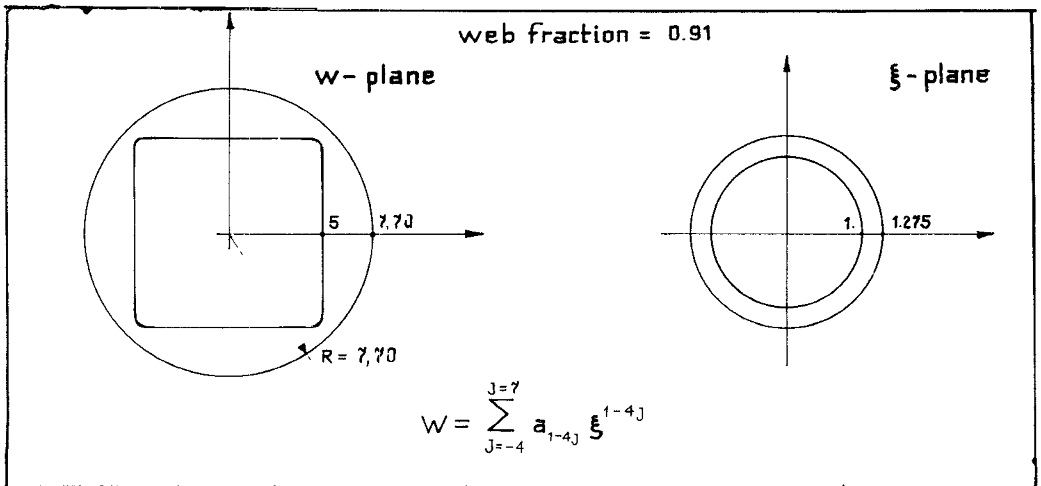
Note: See (2) [7].

where

$$aj = \binom{-\frac{2}{p}}{j} \frac{1}{jp+1} (-1)^j. \tag{5}$$

For instance, in the case of a square domain the mapping function is

$$w = 1.078 a_p \int_0^{\xi} \frac{du}{(1 + u^4)^{1/2}}$$
$$= a_p(1.078\xi - 0.108\xi^5 + 0.045\xi^9 - 0.026\xi^{13} + \cdots). \tag{6}$$



coefficients a_{1-4J}	a_9	a_5	a_1	a_{-3}	a_{-7}	a_{-11}	a_{-15}	a_{-19}	a_{-23}	a_{-27}
	0 0088086	0.1665126	5.9712317	-1 1506136	+0.0159568	-0 0157312	0 0036515	-0 0003575	-0.00021767 6	0 00008544

Fig. 3. Conformal mapping of a doubly connected domain of large web fraction [8].

For $r = 0.10$ the maximum deviations with respect to a perfect circle of radius equal to 0.108 are of the order of $(10^{-6}a_p)$. For $r = 0.75$ the maximum error is approximately equal to 3 percent. If R is the radius of the inner boundary in the w plane (Fig. 1), the corresponding value of the polar coordinate in the ξ plane is given to a good degree of approximation by

$$\lambda = R/a_0' \tag{7}$$

where a_0' is the first coefficient of the mapping polynomial (4).

It will now be shown that if the mapping function for the infinite plane with a polygonal hole is known, it is also possible to map a finite doubly connected domain with an outer circular boundary and an inner polygonal perforation.

Let the function which maps the infinite plane with a polygonal hole onto the infinite plane with a circular hole of radius equal to unity be expressed in terms of an infinite series of the form [4]

$$w = \sum_{n=0}^{\infty} a_{1-np}\xi^{1-np} \tag{8}$$

where p is the number of axes of symmetry of the configuration.

From a practical viewpoint one uses a truncated version of (8). One then has

$$w = \sum_{n=0}^{N} a_{1-np}\xi^{1-np}. \tag{9}$$

Now take $r = \lambda$, where $\lambda \gg 1$, in (9). It is obvious that the first term will be predominant since the remaining terms involve negative powers of the radial variable. In other words, circles in the ξ plane will map into approximate circles of radius R in the w plane. As r approaches unity, the deviations become quite noticeable and a Laurent-type expansion must be used [4]; the analysis and determination of the mapping function requires solution of a system of coupled integral equations [4], [5].

Fig. 2 illustrates the approximations involved in the case of a square inner domain ($2a \times 2a$) when (9) is used. Equation (9) is now expressed by the polynomial

$$w = a(1.1804\xi - 0.1966\xi^{-3} + 0.0214\xi^{-7} - 0.0076\xi^{-11} + 0.0044\xi^{-15} - 0.0026\xi^{-19} + 0.0008\xi^{-23}). \tag{10}$$

When the ratio of radii in the ξ plane is close to unity, the outer boundary deviates considerably from a circle in the w plane. When $R/a = 1.88$, for example, the following truncated Laurent expansion must be used [5] (see Fig. 3):

$$w = a(1.18075474\xi - 0.1991339\xi^{-3} + 0.01719446\xi^{-7} - 0.0049871\xi^{-11} + 0.0001873\xi^{-15} - 0.000745\xi^{-19} + 0.00019\xi^{-23} + 0.0048269\xi^{5} + 0.0000109\xi^{9}). \tag{11}$$

It is important to point out that a very simple approximate procedure is available for the determination of the mapping function for the infinite plane with a hole of regular polygonal shape [6].

The approach discussed in this letter has also been used [8] in the analysis of heat flow and other diffusion-type problems through the walls of long, hollow, prismatic bars (for instance, bars of square cross sections with circular, concentric perforations).[1]

Comparison of the calculated results with those obtained by means of the finite element method show excellent accuracy.

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[1] These configurations are commonly used in nuclear reactor technology.